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# Thermal Stability Analysis of the Darcy-Benard Problem under Mechanical Vibration

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## Abstract

This paper focuses on different standpoints regarding the thermal stability of a horizontal porous layer under the action of vertical vibration. Different time scales are distinguished (i.e. vibrational, viscous, conductive and buoyancy). For the case in which the vibrational time-scale is much smaller than other time scales and the amplitude of vibration is smaller than the height of the layer, the so-called time-averaged method is adopted. The linear stability analysis reveals that the response of the system in this case is harmonic and vibration has a stabilizing effect. Weakly non- linear stability analysis reveals that bifurcation is of the supercritical pitchfork type and an expression for the Nusselt number has been obtained. Although the time-averaged method has the advantage of simplifying the governing equations and providing closed form expression for the optimum choice of control parameters, it fails to describe sub-harmonic modes. In the follow up, it is shown that the stability analysis under arbitrary time-scale relations, called the direct method, leads to the study of a second order parametric oscillator (Mathieu equation). The harmonic and sub-harmonic responses are distinguished. We have put forward a set of conditions from which the stability results obtained by the direct method may be compared with those of the time-averaged method. The validity of the time- averaged method for thermos-vibrational problem has been proved for the first time in porous media. The criterion for the onset of convection has been obtained and this result has been generalized. The behavior of sub-harmonic solution is emphasized. It is shown that at very high frequency, the onset of convection corresponding to this mode does not depend on frequency and is independent of gravity. The response of the solution will be in sub-harmonic mode.

## Introduction

The analysis of buoyancy induced fluid flow and heat transfer in porous media is required in a large number of applications in the fields of heat exchangers, insulating materials, geophysical flows, bio-convection, oil exploration, metal casting, magnetic induced convection, etc. For a review of these topics, see for example Nield and Bejan [1], Ingham and Pop [2] and the handbook of porous media edited by Vafai [3].

Construction of the International Space Station has revived interest in the influence of residual accelerations on convective phenomena. These accelerations could be the result of atmospheric drag, crew activities (inside or outside), the avionics fan or other life supporting machinery on board the space station, which can be modeled as periodical oscillations [4, 5]. Under the action of these oscillations, the driving force is modified.

The active control of convective motion by vibration has been an extraordinary large and vigorous area of research. The study of this class of convective motion from the stability point of view began simultaneously and independently in the East and West through different routes. In the first approach, beginning in the early 70s, which is called the direct method, linear stability is applied to the original system of equations [6-9], where in the momentum equation the buoyancy term contains a periodic time dependent contribution. The resulting system is usually transformed into linear differential equations having periodic coefficients. In the second approach, the so-called time-averaged method, which is valid under the limiting situation of high-frequency and small-amplitude vibration, is used to study the stability of an equilibrium state in fluids [10].

In problems concerning thermal vibration in porous media, Zenkovskaya [11] studies the effect of vertical vibration (parallel to the temperature gradient) on the stability of the horizontal porous layer with infinite horizontal extension. She uses a momentum equation where macroscopic convective terms are included. In her follow up, after performing linear stability analysis these terms are discarded. The resulting system depends on filtration Rayleigh and vibrational Rayleigh numbers. Zenkovskaya and Rogovenko [12] also consider the same problem under various directions of vibration. The imposed vibration has high frequency and small amplitude, which justifies the adoption of the time-averaged formulation. The results of linear stability analysis show that only the vertical vibration always has a stabilizing effect on the onset of convection. These authors find that for other directions of vibration, depending on the vibrational parameter and the vibrational direction, stabilizing or destabilizing effects on the equilibrium solution is possible. Bardan and Mojtabi [13] consider the effect of high-frequency and small-amplitude vibration on the thermal stability of a confined cavity heated from below. The direction of vibration is parallel to the temperature gradient. Their results show that vibration increases the stability threshold and from a weakly nonlinear stability analysis the type of bifurcation is determined.

In contrast to the time-averaged method, which has been used in porous media under the effect of mechanical vibration, the direct method has not received much attention. Khallouf et al [14] study the finite frequency thermo-vibrational convection in a differentially heated square cavity. The direction of vibration is perpendicular to the temperature gradient. The Darcy-Boussinesq model is used and the original system of equations is solved numerically using a pseudo-spectral Chebyshev collocation method. The governing system of equations depends on filtration Rayleigh number, vibrational Rayleigh number and frequency of vibration. Their results show that high-frequency vibration has no effect on convective motion, which is thought to be a consequence of neglecting the time dependent term in the momentum equation.

The aim of the present paper is to compare different approaches (time-averaged and direct) regarding thermal stability of the Darcy-Bénard problem under the effect of vertical vibration. In section (II), the mathematical formulation for obtaining time-averaged governing equations is presented. Based on scale analysis method, different time scales and amplitude ratios are distinguished. The assumptions necessary to find the governing equations are explained. In section (III) the linear stability analysis is outlined in two subsections. In subsection A, the linear stability analysis for time-averaged system of equations is performed. An analytical relationship giving the critical Rayleigh number in terms of vibrational and relevant porous media parameters is presented. The possibility of the onset of thermo- vibrational convection in the presence of gravity under heating condition from below or above is discussed. The case of convective motion under micro-gravity conditions is then studied. In subsection B, the linear stability analysis by direct method (arbitrary frequency) is considered. For different system responses (harmonic or sub-harmonic), the stability of convective motion is determined numerically. In section (IV), it is shown how, by applying certain criteria regarding time scales and amplitude ratios, the two methods can give identical results regarding stability analysis. The validity of time-averaged approach is proved based on a new procedure. An approximate expression for the onset of convection under sub-harmonic response at very high frequency is presented and a new vibrational Rayleigh number is defined. In section (V) a weakly nonlinear stability analysis is presented and the type of bifurcation is determined. An expression for the Nusselt number is also presented. Finally in section (VI), the results are summarized and discussed.

## Mathematical Formulation

The geometry of the problem is shown in Fig. 1; it consists of two horizontal parallel plates having infinite extension. These plates are rigid and impermeable; they are kept at constant but different temperatures $T_1$ and $T_2$. The distance between the plates is $H$. The porosity and permeability of the porous material filling the layer are $\varepsilon$ and $K$ respectively. The porous layer and its boundaries are subjected to a harmonic vibration. We suppose that the porous medium is homogenous and isotropic. The fluid is assumed to be Newtonian and to satisfy the Oberbeck-Boussinesq approximation. In a coordinate system linked to the layer, the gravitational field may be replaced by the sum of the gravitational and vibrational accelerations $\mathbf{g} \rightarrow \mathbf{g} + b\omega^2 \sin(\omega\, t)\mathbf{k}$ where $\mathbf{k}$ is the unit vector along the axis of vibration, $b$ is the displacement amplitude and ω is the angular frequency of vibration. After making standard assumptions (local thermal equilibrium, negligible viscous heating dissipations, …), the governing equations for vertical vibration (parallel to the temperature gradient) are written as:

$$\begin{cases} \nabla \cdot \mathbf{V} = 0, \\ B\dfrac{\partial \mathbf{V}}{\partial t} = -\nabla P + Ra_T T\,(1 + \delta Fr_F \omega^2 \sin \omega t)\mathbf{k} - \mathbf{V}, \\ \dfrac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = \nabla^2 T. \end{cases} \qquad (1)$$

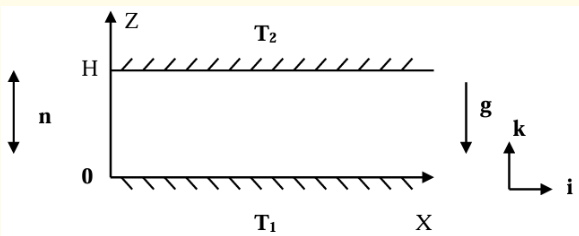


***Figure 1:*** Geometry of the problem.

The boundary conditions corresponding to (1) are:

$$\begin{aligned} &V_z(x, z=0) = 0, \quad T(x, z=0) = 1, \\ &V_z(x, z=1) = 0, \quad T(x, z=1) = 0. \end{aligned} \qquad (2)$$

In system of equations (1), $\mu_f$ is the dynamic viscosity of fluid, $(\rho c)^*$ the effective volumic heat capacity, $(\rho c)_f$ is the volumic heat apacity of fluid and $\lambda^*$ represents the effective thermal conductivity. At this stage some explanations are necessary regarding the use of the Darcy formulation. Although we can include inertia effects (both advective inertia and form-drag), due to the existence of motionless basic state their inclusion does not affect the onset of convection.

### *Time-averaged formulation*

In order to study the averaged behavior of the system (1)-(2), we use the time- averaged method. This method has been used under the conditions of high-frequency and small-amplitude of vibration. Under these conditions which should be carefully defined, it is possible to subdivide the fields into two different parts; the first varies slowly with time (i.e. the characteristic time is long with respect to vibration period) while the second part varies rapidly with time and is periodic with period 2π/ω (this procedure was first used in problems concerning fluid media under vibrational effect by Simonenko and Zenkovskaya [15] and its mathematical justification for the fluid layer under the action of vibration is given by Simonenko [16]).

$$\begin{cases} \mathbf{V}(M,t)=\overline{\mathbf{V}}(M,t)+\mathbf{V}'(M,\omega t), \\ T(M,t)=\overline{T}(M,t)+T'(M,\omega t), \\ P(M,t)=\overline{P}(M,t)+P'(M,\omega t). \end{cases} \tag{3}$$

In (3), $(\overline{\mathbf{V}},\overline{T},\overline{P})$ represent the averaged fields (for a given function f $(M, t)$, the average is defined as $\bar{f}(M,t)=\frac{1}{\tau}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} f(M,s)\,ds$).

On replacing (3) into equations (1)-(2), and by performing averaging procedures over a vibration period and separating the oscillatory fields from mean fields two coupled systems of equations are obtained. One governs the mean flow and the other the oscillatory flow.

The governing equations for the mean flow are:

$$\begin{cases} \nabla\cdot\overline{\mathbf{V}}=0, \\ B\dfrac{\partial\overline{\mathbf{V}}}{\partial t}=-\nabla\overline{P}+Ra_T\overline{T}\,\mathbf{k}+Ra_T\delta Fr_F\omega^2\overline{T'\sin\omega t}\,\mathbf{k}-\overline{\mathbf{V}}, \\ \dfrac{\partial\overline{T}}{\partial t}+\overline{\mathbf{V}}\cdot\nabla\overline{T}+\overline{\mathbf{V}'\cdot\nabla T'}=\nabla^2\overline{T}. \end{cases} \tag{4}$$

The governing equation for the oscillatory flow can then be written as:

$$\begin{cases} \nabla\cdot\mathbf{V}'=0, \\ B\dfrac{\partial\mathbf{V}'}{\partial t}=-\nabla P'+Ra_T\delta Fr_F\omega^2\overline{T}\,\sin\omega t\,\mathbf{k} \\ \qquad +Ra_T T'\,(1+\delta Fr_F\omega^2\sin\omega t)\mathbf{k}-Ra_T\delta\omega^2 Fr_F\overline{T'\sin\omega t}\,\mathbf{k}-\mathbf{V}', \\ \dfrac{\partial T'}{\partial t}+\mathbf{V}'\cdot\nabla\overline{T}+\mathbf{V}'\cdot\nabla T'+\overline{\mathbf{V}}\cdot\nabla T'-\overline{\mathbf{V}'\cdot\nabla T'}=\nabla^2 T'. \end{cases} \tag{5}$$

It can be observed that systems of equations (4) and (5) are coupled due to the terms $\overline{T'\sin\omega t}$ and $\overline{V'.\nabla T'}$ obtained from time-averaging procedure.

The next step is to simplify (5) as much as possible with the objective to express the oscillating fields in terms of averaged ones. Before proceeding with a discussion on the time- averaged method, it is informative to describe the nature of momentum and energy equations in the thermo-vibrational problem. For this reason we perform an order magnitude analysis. We emphasize that due to complexity of the problem, if the analysis from non-dimensional equations is begun, erroneous results may be encountered.

### *Scale analysis method for the oscillatory system*

The key step in resolving the closure problem lies in establishing relations between oscillatory velocity and temperature fields in terms of averaged ones. For this purpose, the scale analysis method is used. This method has been successfully employed by Bejan [17-18] in predicting boundary layer approximations, the existence of optimal geometries and critical parameters. An introductory discussion of Davis [19] on the importance of scale analysis in the context of time-periodic flows is highly recommended. The following scales are used in the oscillatory system of equations:

$$O(\overline{T})\approx 1,\quad O(\frac{\partial(\quad)}{\partial t})\approx\omega(\quad), \tag{6}$$

By replacing these scales in the oscillating momentum equation and assuming that for the oscillating temperature scale $T'<(\overline{T}\approx 1)$, we find the order of magnitude of important terms in (5):

inertia: $O(B\frac{\partial \mathbf{V}'}{\partial t}) \approx B\text{v}'_{\text{scale}}\omega$, buoyancy: $O(Ra_T\overline{T}\delta Fr_F\omega^2 \sin \omega t) \approx Ra_T\delta Fr_F\omega^2$ and friction: $O(\mathbf{V}') \approx \text{v}'_{\text{scale}}$.

The buoyancy terms involving $T'$ may be neglected (the condition for this assumption will be explained later).

In order to study the possibility of convective motion in the oscillatory momentum equation, we consider the following case:

$$\text{Buoyancy} \approx \text{Inertia} \tag{7}$$

It should be emphasized that given the fact that our problem is buoyancy driven flows, the buoyancy term is the driving force of the flow.

A necessary final step is to determine the domain in which the results are valid, so we may write:

$$\text{Inertia} >> \text{Friction} \tag{8}$$

By replacing the order magnitudes of corresponding terms in expression (7), the oscillating velocity scale may be obtained:

$$\text{v}'_{\text{scale}} \approx \frac{Ra_T\delta Fr_F\omega}{B} \tag{9}$$

Furthermore, from the inequality (8) we get:

$$1 < \omega \text{ or } \tau_{vib} << \tau_{visc} \tag{10}$$

In relation (10) $\tau_{vib} = 1/\omega$ and $\tau_{visc} = K/\varepsilon\nu$ which represent vibrational and viscous time scales, respectively ($\nu$ is the kinematic viscosity). Assumption (10) allows us to neglect the viscous term in the oscillating momentum equation.

Following the same procedure, the order magnitude of important terms in oscillatory energy equation is found:

Transient term: $\frac{\partial T'}{\partial t} \approx T'\omega$, Convection: $\mathbf{V}'\cdot\nabla\overline{T} \approx v'_{scale}$, and Diffusion: $\nabla^2 T' \approx T'$.

To study the possibility of oscillatory convective motion the following case is studied:

$$\text{Convection} \approx \text{Inertia} \tag{11}$$

Again the domain of validity is as follows:

$$\text{Transient term} >> \text{Diffusion} \tag{12}$$

Imposing the velocity scale (9) in the equality (11) and using the hypothesis $T' << \overline{T}(\approx 1)$ results in:

$$T' \approx \frac{Ra_T\delta Fr}{B} \qquad (\sigma = \frac{(\rho c)_*}{(\rho c)_f}) \tag{13}$$

Furthermore, from the hypothesis T' < 1 and (13) we obtain:

$$\frac{Ra_T\delta Fr}{B} < 1 \tag{14}$$

Inequality (14) gives the criterion for small-amplitude vibration. Also, from (12), we obtain:

$$\omega > 1 \quad \text{or} \qquad \tau_{vib} << \tau_{cond}, \quad (a_* = \frac{\lambda_*}{(\rho c)_f}) \tag{15}$$

In relation (15) $\tau_{cond} = \sigma H^2/a_*$ represents conductive time scale. Relation (15) allows us to neglect the diffusive term in the oscillatory energy equation.

Now that the scale of $T'$ has been found, the final step is to validate our assumptions in the oscillatory momentum equation; in other words it should be shown under which condition $\rho_0\beta_T\Delta Tb\omega^2$ is the dominant buoyancy force. In order to answer this question all the buoyancy forces in the oscillatory momentum equation are represented by their order of magnitude equivalence in table 1. A close look at this table reveals that if (I) >> (II), so $\rho_0\beta_T\Delta Tb\omega^2$ is the dominant buoyancy force in the oscillatory momentum equation, from which we conclude:

$$\omega^2 >> \frac{Ra_T}{B} \quad \text{or} \quad \tau_{vib}^2 << \tau_{buoy}^2. \tag{16}$$

| *Different buoyancy terms in oscillatory momentum equation* | *Order of magnitude representation* |
|---|---|
| I. $\rho_0\beta_T(\overline{T} - T_2)b\omega^2 \sin\omega t$ | $\rho_0\beta_T\Delta Tb\omega^2$ |
| II. $\rho_0\beta_T T'g$ | $\rho_0\beta_T T'g$ |
| III. $\rho_0\beta_T T'b\omega^2 \sin\omega t$ | $\rho_0\beta_T T'b\omega^2$ |
| IV. $\overline{\rho_0\beta_T T'b\omega^2 \sin\omega t}$ | $\rho_0\beta_T T'b\omega^2$ |

***Table 1:*** Order magnitude analysis of different buoyancy terms.

In (16), the gravitational buoyancy time scale is defined as $\tau_{buoy} = (\varepsilon g\beta_T\Delta T/\sigma H)^{1/2}$.

Inequality (16) along with (15) and (10) constitute the frequency range for achieving high- frequency vibration. It should be noted that, to our knowledge, no systematic scale analysis method has been reported in thermo-vibrational problems in porous media. In table 2 the assumptions necessary to obtain closure in the oscillatory system in porous media are compared to those in fluid media.

| *Fluid System* | *Porous System* |
|---|---|
| $\tau_{vib} << min(\frac{H^2}{\nu}, \frac{H^2}{a})$ | $\tau_{vib} << min(\frac{K}{\varepsilon\nu}, \frac{\sigma H^2}{a_*})$ |
| $b << \frac{H}{\beta_T\Delta T}$ | $b << \frac{H}{\frac{\varepsilon}{\sigma}\beta_T\Delta T}$ |
| $\omega^2 >> \frac{g}{H}(\beta_T\Delta T)$ | $\omega^2 >> \frac{g}{H}\cdot\frac{\varepsilon\beta_T\Delta T}{\sigma}$ |

***Table 2:*** Comparison of assumptions for finding the time-averaged formulation in fluid and porous system of equations.

### *Time-averaged system of equations*

By applying assumptions (10), (14), (15), (16) to the oscillatory system of equations may be simplified significantly:

$$\begin{cases} B\dfrac{\partial \mathbf{V}'}{\partial t} = -\nabla P' + Ra_T \overline{T} \delta Fr_F \omega^2 \sin \omega t\, \mathbf{k} \\ \dfrac{\partial T'}{\partial t} + \mathbf{V}' \cdot \nabla \overline{T} = 0 \end{cases} \quad (17a)$$

By using Helmholtz's decomposition, we may eliminate the oscillatory pressure. ($\boldsymbol{W}$ and $\nabla\boldsymbol{\varphi}$ are solenoidal and irrotational parts of Helmholtz's decomposition):

$$\overline{T}\,\mathbf{k} = \mathbf{W} + \nabla \varphi. \quad (17b)$$

This allows the finding of oscillatory velocity and temperature:

$$\begin{cases} \mathbf{V}' = -(\dfrac{Ra_T \delta Fr_F \omega}{B} \cos \omega t)\mathbf{W}, & (18) \\ T' = (\dfrac{Ra_T \delta Fr_F}{B} \sin \omega t)\mathbf{W} \cdot \nabla \overline{T}. & (19) \end{cases}$$

By substituting Eqs. (18) and (19) in the coupling terms of mean fields (4), the time-averaged equations are found. We introduce reference parameter, $T_1$-$T_2$ for temperature, $H$ for height, $\sigma H^2/a^*$ for time, $a^*/H$ for velocity, $\beta_T \Delta T$ for W and $\mu a^*/K$ for pressure. The resulting averaged system in dimensionless form may be written as:

$$\begin{cases} \nabla \cdot \overline{\mathbf{V}}^* = 0, \\ \overline{\mathbf{V}}^* = -\nabla \overline{P}^* + Ra_T \overline{T}^* \mathbf{k} + Ra_v\, (\mathbf{W}^* \cdot \nabla) \overline{T}^* \mathbf{k}, \\ \dfrac{\partial \overline{T}^*}{\partial t} + \overline{\mathbf{V}}^* \cdot \nabla \overline{T}^* = \nabla^2 \overline{T}^*, \\ \nabla \cdot \mathbf{W}^* = 0, \\ \nabla \times \mathbf{W}^* = \nabla \overline{T}^* \times \mathbf{k}. \end{cases} \quad (20)$$

The corresponding boundary conditions for this system are:

$$\begin{aligned} &\forall x^*,\ for\ z^* = 0, \quad \overline{V}_z^* = 0, \quad \overline{T}^* = 1, \quad W_z^* = 0, \\ &\forall x^*,\ for\ z^* = 1, \quad \overline{V}_z^* = 0, \quad \overline{T}^* = 0, \quad W_z^* = 0. \end{aligned} \quad (21)$$

in which:

$$Ra_T = \frac{\mathrm{K} g \beta_T \Delta T H}{\nu a_*}, \quad Ra_v = \frac{(\delta^* Fr_F Ra_T \omega^*)^2}{2B},$$

$$(\delta^* = \frac{b}{H},\ Fr_F = \frac{a_*^2}{gH^3\sigma^2},\ \omega^* = \omega\frac{\sigma H^2}{a_*},\ B = \frac{a_* \mathrm{K}}{\varepsilon \nu \sigma H^2} = \frac{\tau_{visc}}{\tau_{cond}})$$

In the above relationships $Ra_T$ is the thermal Rayleigh number, $Ra_v$ is the vibrational Rayleigh number, $\omega^*$ is the dimensionless frequency, $B$ is the ratio viscous time scale to conductive time scale, $Fr_F$ is the filtration Froude number and $\delta^*$ is the dimensionless amplitude.

It is important to note that in (20), the transient term in momentum equation due to frequency consideration has been neglected.

## Stability Analysis

### *Linear stability analysis under high frequency and small amplitude vibration (time- averaged formulation)*

The steady-state distribution of velocity, temperature and solenoidal fields are sought. The equilibrium state corresponds to a linear distribution for temperature and zero for solenoidal and velocity fields:

$$\overline{T}_0^* = 1 - z^*, \quad \mathbf{W}_0^* = 0. \quad (22)$$

For linear stability analysis, the temperature, velocity, pressure and solenoidal fields are perturbed around the equilibrium state (for simplicity bars are omitted):

$$\mathbf{V}^* = 0 + \mathbf{v}', T^* = T_0^* + T', P^* = P_0^* + p', \mathbf{W}^* = \mathbf{W}_0^* + \mathbf{w}'.$$

Replacing the above equations in system (20,21), and after performing the standard linearization procedure we obtain:

$$\begin{cases} \nabla \cdot v' = 0, \\ v' = -\nabla p' + Ra_T T' \mathrm{k} + Ra_v (\mathrm{w}' \cdot \nabla T_0^* + \mathrm{W}_0^* \cdot \nabla T') \mathrm{k}, \\ \dfrac{\partial T'}{\partial t^*} + v' \cdot \nabla T_0^* = \nabla^2 T', \\ \nabla \cdot \mathrm{w}' = 0, \\ \nabla \times \mathrm{w}' = \nabla T' \times \mathrm{k}. \end{cases} \quad (23)$$

with corresponding boundary conditions:

$$\begin{aligned} &\mathrm{v}'_z(x^*, z^* = 0) = 0,\ T'(x^*, z^* = 0) = 0,\ \mathrm{w}'_z(x^*, z^* = 0) = 0, \\ &\mathrm{v}'_z(x^*, z^* = 1) = 0,\ T'(x^*, z^* = 1) = 0,\ \mathrm{w}'_z(x^*, z^* = 1) = 0. \end{aligned} \quad (24)$$

Introducing $\Psi$ and $F$ for velocity and solenoidal field disturbances we can write:

$$\mathrm{v}'_x = \frac{\partial \psi}{\partial z^*},\ \mathrm{v}'_z = -\frac{\partial \psi}{\partial x^*},\ \mathrm{w}'_x = \frac{\partial F}{\partial z^*},\ \mathrm{w}'_z = -\frac{\partial F}{\partial x^*}. \quad (25)$$

2D disturbances are developed in normal forms:

$$(\psi, T', F) = (\phi(z^*), \theta(z^*), f(z^*)) \exp(-\lambda t^* + ikx^*) \quad (26)$$

Replacing (26) in (23)-(24) and eliminating the pressure we obtain:

$$\begin{cases} \dfrac{d^2\phi(z^*)}{dz^{*2}} - k^2\phi(z^*) = -ikRa_T\theta(z^*) + k^2 Ra_v f(z^*), \\ -\lambda\theta(z^*) + ik\phi(z^*) = \dfrac{d^2\theta(z^*)}{dz^{*2}} - k^2\theta(z^*), \\ -k^2 f(z^*) + \dfrac{d^2 f(z^*)}{dz^{*2}} = -ik\theta(z^*). \end{cases} \quad (27)$$

In (27) $k$ is the wave number and $\varphi, \theta, f$ represent amplitude of velocity, solenoidal and temperature disturbances, respectively. Also, $\lambda$ characterizes the eigenvalue of the system, which is generally a complex number ($\lambda = \lambda_r + i\lambda_i$).

$$\lambda = \lambda(Ra_T, Ra_v, k, B)$$

There exist exact solutions of sinusoidal form:

$$(\phi(z^*), \theta(z^*), f(z^*)) = (\phi, \theta, f) \sin n\pi z^*$$

which upon replacing in (27) results in the following relation for marginal stability ($\lambda$=0):

$$Ra_T = \frac{(\pi^2 + k^2)^2}{k^2} + Ra_v \frac{k^2}{\pi^2 + k^2} \cdot \quad (28)$$

For all values of control parameters, it has been verified numerically that $\lambda_i = 0$. It can be understood from the above equation that, under micro-gravity ($Ra_T = 0$), the system remains thermally stable. Under the condition of vibration in the presence of gravity, $Ra_v$ can be replaced with $(\delta^* FrF\omega^* Ra_T)^2/2B$. After some arrangement we obtain the following relations for critical values ($dRa_T/dk^2 = 0$):

$$\begin{cases} Ra_{Tc} = \dfrac{(2\pi^2 - k_c^2)(\pi^2 + k_c^2)^2}{\pi^2 k_c^2} \\ \dfrac{(\delta^* Fr_F \omega^*)^2}{2B} = \dfrac{\pi^2(\pi^2 - k_c^2)}{(2\pi^2 - k_c^2)^2(\pi^2 + k_c^2)} \end{cases} \quad (29)$$

Fig. 2, illustrates that for a porous layer heated from below, vibration significantly increases the stability limit, this effect is more pronounced at higher frequencies. At the same time vibration modifies the convection patterns; it reduces the wave number. Another important fact is the layer under the situation heated from above is thermally stable.

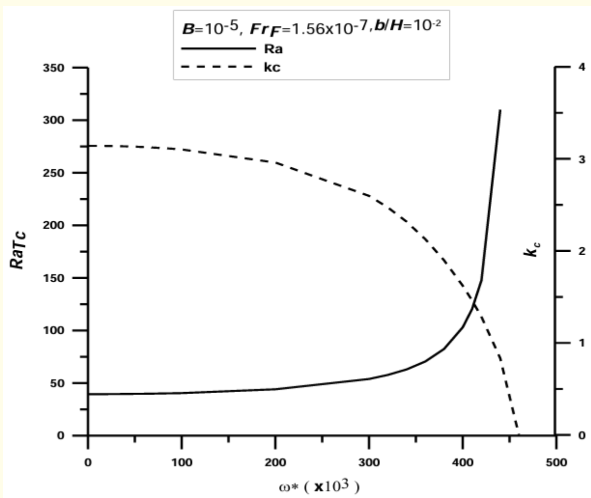


***Figure 2:*** The effect of vibration on the critical thermal Rayleigh $Ra_{Tc}$ and wave numbers $k_c$ as a function of dimensionless frequency $\omega^*$ (time-averaged formulation).

In order to provide some ideas about the effect of vibration, numerical simulations for confined cavity (H/L = 4) have been conducted, using commercial code Femlab. As is evident from Fig. 3, vibration reduces the convective rolls.

Another interesting feature of equation (29) is that we may obtain the maximum limit of time-averaged method:

$$\omega^*_{\max} = \frac{\sqrt{B/2}}{\delta^* Fr_F \pi} \cdot \qquad (k_c \to 0) \quad (30)$$

We should emphasize that this limit was obtained due to redefinition of $Ra_v$, this important point has been ignored by several authors who adopted time-averaged formulation.

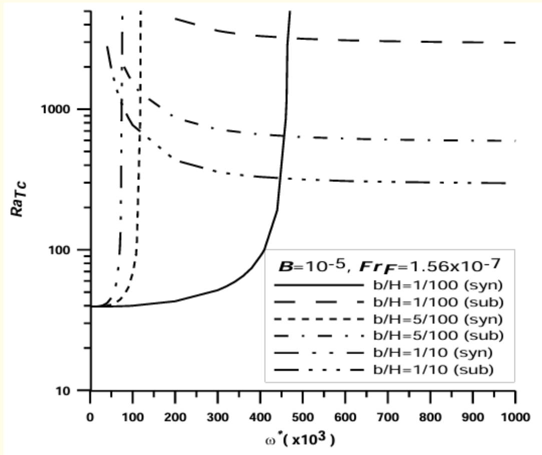


***Figure 3:*** The effect of vibration on the critical Rayleigh number $Ra_{Tc}$ for the layer heated from below as a function of the dimensionless $\omega^*$ for different values of dimensionless amplitude *b/H* for harmonic (synchronous) and sub-harmonic modes.

An interesting question at this stage may be put forward: what would happen if this frequency is exceeded, in other words, would the layer remain thermally/ linearly stable or not? From (18)-(19), it can be seen that the response of the system is harmonic and from (30) it can be observed that this limit cannot be exceeded. So, in order to respond to this question, the problem in the context of arbitrary frequency should be studied.

### *Linear stability analysis under arbitrary frequency of vibration (direct method)*

In this part, the thermal stability of the governing equations in the original form is examined. When the direction of vibration is parallel to the temperature gradient, mechanical equilibrium is possible which is characterized by linear temperature and parabolic pressure distributions (refer to (1) and (2)). In order to study linear stability, the velocity and temperature fields are infinitesimally perturbed around the motionless equilibrium state. The perturbed system becomes:

$$\begin{aligned} &\nabla\cdot\tilde{\mathbf{v}}=0,\\ &B\frac{\partial\tilde{\mathbf{v}}}{\partial t}=-\nabla\tilde{p}+Ra_T\tilde{\theta}\,(1+\delta Fr_F\omega^2\sin\omega t)\,\mathbf{k}-\tilde{\mathbf{v}},\\ &\frac{\partial\tilde{\theta}}{\partial t}+\tilde{\mathbf{v}}\cdot\nabla T_0=\nabla^2\tilde{\theta}. \end{aligned} \tag{31}$$

After eliminating the pressure in (31) and developing the disturbances in the form of normal modes:

$$\tilde{v}_z=X(t)\,e^{ikx}\sin z\pi,\;\;\tilde{\theta}=h(t)e^{ikx}\sin z\pi. \tag{32}$$

thus by defining Y with components as expressed in (32), we obtain the following equation:

$$\frac{dY}{dt}=M_0Y+\sin\omega tN_0Y \tag{33a}$$

with $M_0$ and $N_0$ defined as:

$$M_0 = \begin{bmatrix} -\frac{1}{B} & \frac{Ra_T}{B}\frac{k^2}{k^2+\pi^2} \\ 1 & -(\pi^2+k^2) \end{bmatrix}, \quad N_0 = \begin{bmatrix} 0 & \frac{Ra_T}{B}\delta Fr\omega^2\frac{k^2}{k^2+\pi^2} \\ 0 & 0 \end{bmatrix} \tag{33b}$$

We may arrange (33a) as:

$$B\frac{d^2h}{dt^2} + \left[B(k^2+\pi^2)+1\right]\frac{dh}{dt} + \left[(k^2+\pi^2) - Ra_T\frac{k^2}{k^2+\pi^2}(1+\delta Fr_F\omega^2\sin\omega t)\right]h = 0. \tag{34}$$

Furthermore, by substituting $e^{-mt}M(t)$ in this equation the classical Mathieu equation (m being $(\pi^2 + k^2 + 1/B)/2$) is found):

$$\frac{d^2M(\tau)}{d\tau^2} + (A - 2Q\cos 2\tau)M(\tau) = 0, \quad (\omega t = 2\tau - \frac{\pi}{2}) \tag{35a}$$

where $A$ and $Q$ are defined as:

$$A = \frac{4}{\omega^{*2}}\left[\frac{\pi^2+k^2}{B} - m^2 - \frac{k^2}{B(\pi^2+k^2)}Ra_T\right],. \tag{35b}$$

$$Q = \frac{2k^2}{B(\pi^2+k^2)}\delta^* Fr_F Ra_T \tag{35c}$$

Detailed analysis of the stable regions for this equation for harmonic and sub-harmonic solutions can be found elsewhere [20-22]. For different applications of this equation in industry see for example [23]. In order to solve eq. (35a), the Floquet theory is used, which considers the solution in exponential form; $M = R(\tau)e^{\mu\tau}$, where $R(\tau)$ is a periodic function having period $\pi$ or $2\pi$ and parameter $\mu$ is the Floquet exponent. The marginal stability condition is satisfied when $m = \frac{\mu\omega^*}{2}$. Based on the Floquet theory, the solutions of Mathieu equation (35a) are expressed as double periodic series:

$$M_\pi = e^{\mu\tau}\sum_{-\infty}^{+\infty} a_n e^{2in\tau}, \tag{36a}$$

$$M_{2\pi} = e^{\mu\tau}\sum_{-\infty}^{+\infty} a_n e^{i(2n+1)\tau}. \tag{36b}$$

These correspond to harmonic and sub-harmonic solutions, respectively. Substituting each of these equations in the Mathieu equation results in Hill's determinant.

Replacing the above equations in (35a) (with $\cos\omega^* t^* = \frac{e^{i\omega^* t^*} + e^{-i\omega^* t^*}}{2}$) and after some rearrangements we find a recursive relation [23b]:

$$\xi_n a_{n-1} + a_n + \xi_n a_{n+1} = 0 \quad n = \dots, -2, -1, 0, 1, 2, \dots\dots$$

where $\xi_n(\mu) = \frac{Q}{(2n - i\mu)^2 - A}$. The recursive relation represents an infinite set of homogeneous equations for $\{a_n\}$ having non-zero solutions; this is only true if the infinite determinant $\Delta(i\mu)$ formed by coefficients of recursive relation becomes zero. This determinant may be written as;

$$\cosh(\mu\pi) = 1 - 2\Delta(0)\sin^2(\frac{\pi\sqrt{A}}{2}), \qquad A > 0 \tag{37a}$$

$$cosh(\mu\pi) = 1 + 2\Delta(0)\, sinh^2(\frac{\pi\sqrt{|A|}}{2}). \qquad A<0 \tag{37b}$$

furthermore $\Delta\,(0)$ is defined for $\mu = 0$ and is calculated from another recursive relations:

$$\Delta_0 = 1, \Delta_1 = 1 - 2\xi_0\xi_1, \Delta_2 = (1-\xi_1\xi_2)^2 - 2\xi_0\xi_1(1-\xi_1\xi_2),$$
$$\Delta_{n+2} = (1-\xi_{n+1}\xi_{n+2})\Delta_{n+1} - \xi_{n+1}\xi_{n+2}(1-\xi_{n+1}\xi_{n+2})\Delta_n + \xi_n^2\xi_{n+1}^3\xi_{n+2}\Delta_{n-1}. \tag{38}$$

Following a similar procedure for the solution with period $2\pi$ with applying the transformation $\mu + i = \mu'$, we find the recursive relation (38) and the resulting characteristic equations will become:

$$cosh\,\mu\pi = -1 + 2\Delta(0)\, sin^2(\frac{\pi\sqrt{A}}{2}), \qquad A > 0 \tag{39}$$

$$cosh\,\mu\pi = -1 - 2\Delta(0)\, sinh^2(\frac{\pi\sqrt{|A|}}{2}). \qquad A < 0 \tag{40}$$

The details of the method are given in [24-25] and are not repeated here.

To obtain the critical thermal Rayleigh and wave numbers for marginal stability, all the working parameters ($B$, $\omega^*$, $\delta^*$, $Fr_F$) are fixed except $Ra_T$ and $k$. A search is then undertaken for obtaining the minimum of $Ra_T$ as a function of $k$, the results are shown in Figs. 4 to 9. It can be observed that for a given set of variables ($B$, $\omega^*$, $\delta^*$, $Fr_F$), there are two modes of convective onset, namely harmonic and sub-harmonic. The computations are realized for heating from below ($Ra_T > 0$) and heating from above ($Ra_T < 0$).

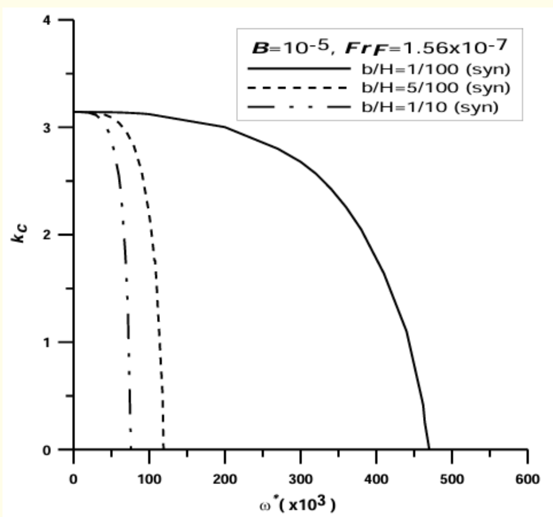


***Figure 4:*** The effect of vibration on critical wave number ($k_c$) as a function of dimensionless frequency $\omega^*$ for harmonic (synchronous) solutions for different dimensionless amplitude with direct formulation (for the layer heated from below).

When the layer is heated from below, harmonic and sub-harmonic responses are possible. These responses demonstrate different behaviors. An important point should be noted at this stage, which concerns the appearance of these modes. The necessary condition for obtaining these responses, which has been neglected is that $B\omega^* \approx 1$ (see Eq. 33), otherwise we cannot obtain a second order differential equation.

For harmonic mode with increasing $\omega^*$, the system becomes thermally more stable which causes $Ra_{Tc}$ to increase. The extension of harmonic domain depends significantly on the choice of dimensionless amplitude *b/H*, Fig. 4. From the pattern formation point of view, with increasing $\omega^*$ critical wave number $k_c$ decreases; $k_c$ is also highly sensitive to *b/H*, Fig. 5. For sub-harmonic mode there is a different scenario, the vibration has a destabilizing effect. It should be emphasized that the reference here is the intersection of the two curves corresponding to harmonic and sub-harmonic modes, however if $4\pi^2$ is considered as the reference, it appears to have a stabilizing effect. In this case, $Ra_{Tc}$ decreases and ultimately reaches a limiting value independent of frequency. This point will be discussed in the next section. The critical wave number for this mode increases with $\omega^*$; b/H has no effect on critical wave numbers at higher frequencies, Fig.6. If we increase $\delta^*$, it can be observed that the intersection of harmonic and sub-harmonic modes shift towards lower frequencies, Fig. 4.

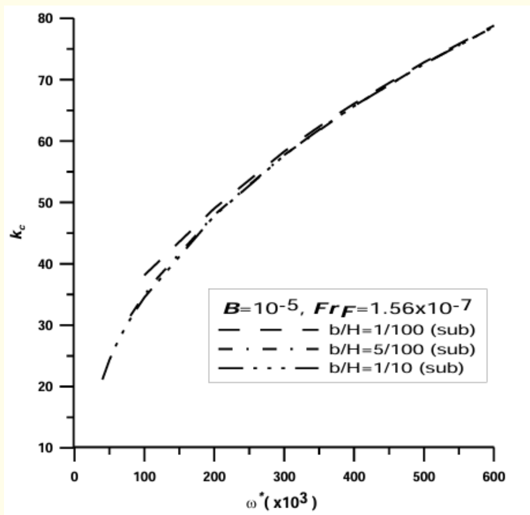


***Figure 5:*** The effect of vibration on critical wave number ($k_c$) as function of dimensionless frequency $\omega^*$ for sub-harmonic solution for different dimensionless amplitude with direct formulation (heating from below).

When the layer is heated from above, under the condition explained above, the onset of convection is possible for both harmonic and sub-harmonic responses. In both of these modes, with increasing dimensionless frequency, the thermal Rayleigh numbers begin from high values and then sharply reduce and finally tend to asymptotic values, Fig. 7. Destabilization depends on *b/H*. It should be noted that the frequency of harmonic mode is twice the frequency of external vibration; it is better to refer to this mode as super-harmonic. This point which is in perfect agreement with conclusions related to Eq.28 has never been reported [6, 7, 9, 25, 25b]. The critical wave number $k_c$ of super-harmonic mode increases rapidly and then tends to a limiting value (c.f. Fig. 8) which is in severe contrast to the behavior of wave number for $Ra_T > 0$. For the sub-harmonic mode, the same behavior similar to the case of heating from below can be observed, Fig.9. The most interesting result of our study for sub- harmonic response is that at very high frequency the values of the critical Rayleigh number tend to an asymptotic value as in the case of heating from below. This means that the gravitational effect is not important, which leads to the conclusion that under micro-gravity conditions the onset of convection may be in sub-harmonic mode. This will be discussed in the next section.

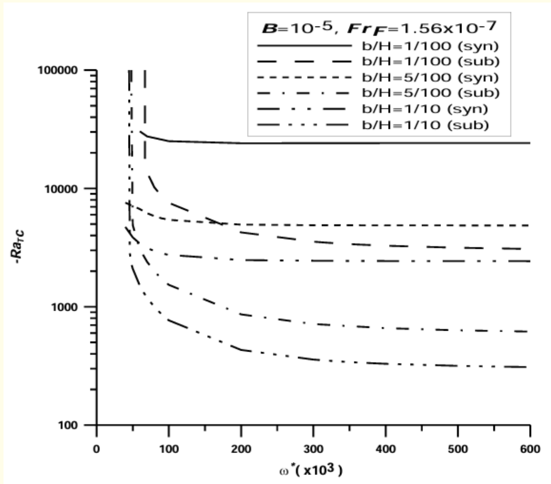


***Figure 6:*** The effect of vibration on the critical Rayleigh number $Ra_{Tc}$ for the layer heated from above as a function of the dimensionless $\omega^*$ for different values of dimensionless amplitude *b/H* for super-harmonic and sub-harmonic modes.

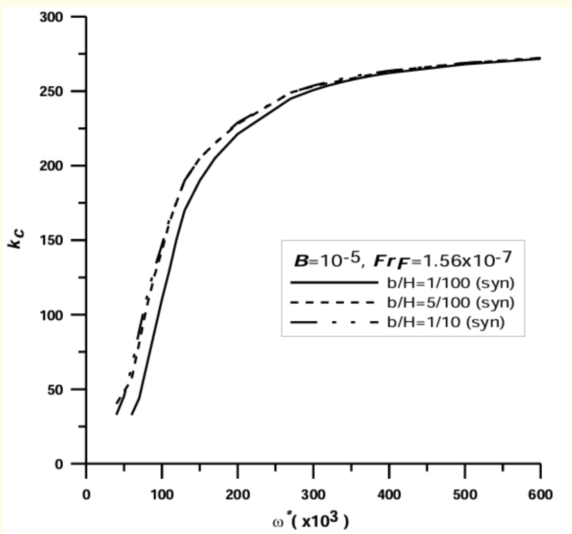


***Figure 7:*** The effect of vibration on critical wave number ($k_c$) as a function of dimensionless frequency $\omega^*$ for super-harmonic solutions (the layer heated from above).

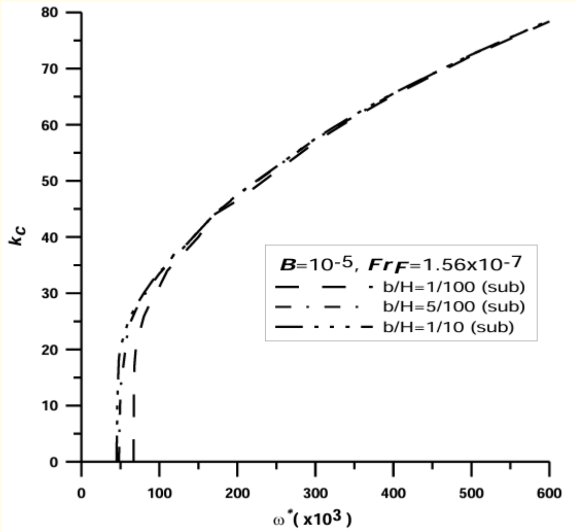


***Figure 8:*** The effect of vibration on critical wave number ($k_c$) as function of dimensionless frequency $\omega^*$ for sub-harmonic solution for different dimensionless amplitude (heated from above).

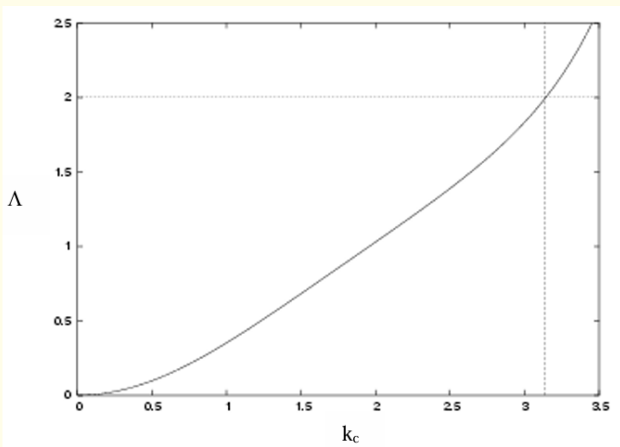


***Figure 9:*** The value of Λ as a function of $k_c$.

## Comparison of the Two Methods

The objective of this section is to compare the two stability analysis approaches used in the thermo-vibrational problem, namely the time-averaged and the direct methods. As shown in the second section of the study, under the time-averaged method, the high frequency and small amplitude situation is considered. This special situation allows us to subdivide the temperature and velocity fields into two parts. The question is, under which condition this subdivision of fields can be obtained if the direct method is adopted. Mathieu's equation coefficients $A$ and $Q$ can be arranged as:

$$A = -\left[\frac{\tau_{vib}}{\tau_{cond}}(k^2+\pi^2) + \frac{\tau_{vib}}{\tau_{visc}}\right]^2 + 4\left(\frac{\tau_{vib}^2}{\tau_{cond}\tau_{visc}}\right)(k^2+\pi^2) - 4\left(\frac{\tau_{vib}}{\tau_{buoy}}\right)^2\frac{k^2}{k^2+\pi^2},$$
$$Q = 2\left(\frac{\varepsilon}{\sigma}\beta_T\Delta T\frac{b}{H}\right)\frac{k^2}{k^2+\pi^2}.$$

Close examination of A and Q reveals the following:

The first and second terms in $A$ involve the two assumptions on conductive and viscous time scales with respect to vibrational time scale (15) and (10) while the third term involves the assumption on the gravitational buoyancy time scale with respect to the vibrational time scale (16). $Q$ involves the hypothesis on small amplitude (14). Based on the hypothesis of high-frequency and small amplitude all these terms are very small, thus $A$ and $Q$ together tend to zero.

A regular perturbation method in which $Q$ is considered as a small parameter is used followed by a search for a stable solution:

$$M(\tau) = M_0(\tau) + QM_1(\tau) + Q^2M_2(\tau) + \cdots$$
$$A = A_0 + QA_1 + Q^2A_2 + \cdots \quad (41)$$

Replacing the above expansions in Mathieu's equation results in following systems:

$$Q^0 : \frac{d^2M_0}{d\tau^2} + A_0M_0 = 0, \quad (42a)$$
$$Q^1 : \frac{d^2M_1}{d\tau^2} + A_0M_1 = -A_1M_0 + 2M_0\cos 2\tau, \quad (42b)$$
$$Q^2 : \frac{d^2M_2}{d\tau^2} + A_0M_2 = -A_2M_0 + 2M_1\cos 2\tau - A_1M_1. \quad (42c)$$

We search for a stable solution ($a_0$ is an arbitrary constant):

$$A_0 = 0 \Rightarrow M_0 = a_0 = constant,$$
$$A_1 = 0 \Rightarrow M_1 = -\frac{a_0}{2}\cos 2\tau.$$

By substituting the above relation in (42c), we get:

$$\frac{d^2M_2}{d\tau^2} = -a_0\left(\frac{1}{2} + A_2\right) - \frac{a_0}{2}\cos 4\tau. \quad (43)$$

The necessary condition for obtaining a stable periodic solution in (43) is to consider:

$$A_2 = -\frac{1}{2}.$$

On replacing $A_0$, $A_1$, $A_2$ in (41) we obtain:

$$A = -\frac{Q^2}{2}, \quad (44a)$$
$$M = a_0 - \frac{a_0}{2}\cos 2\tau. \quad (44b)$$

On replacing $A$ and $Q$ in Eq. (44a) and using the condition for marginal stability $\mu = [a^*(k^2 + \pi^2)/\ \sigma H^2\omega + \varepsilon\nu/K\omega)] = 0$, we find:

$$Ra_T = \frac{(\pi^2+k^2)^2}{k^2} + Ra_v\frac{k^2}{k^2+\pi^2} \quad \left(Ra_v = \frac{(\delta^* Fr_F\omega^* Ra_T)^2}{2B}\right). \quad (45a)$$

This means that imposing the assumptions needed for the averaging method on Mathieu's equation gives identical results to the time-averaged formulation. For comparison the results are compared in table 3. The most interesting aspect is that the time-averaged method gives only harmonic (with dimensionless frequency $\omega^*$) mode and is not able to give a sub- harmonic mode. Another aspect is the generic form of Eq.(45a), it is composed of two parts one is due to the gravitational effect and the other related to vibration. We repeated the same procedure here to the Mathieu equation given in [25], it may be written as:

$$Ra_T = \frac{12(\pi^2 + k^2)^2}{k^2} + Ra_v \frac{k^2}{k^2 + \pi^2} \tag{45b}$$

| | ***Time-Averaged Method*** | | ***Direct Method*** | |
|---|---|---|---|---|
| ***ω*(x103)*** | $Ra_c$ | $k_c$ | $Ra_c$ | $k_c$ |
| 200 | 40.2365 | 3.1114 | 40.2857 | 3.1095 |
| 300 | 41.3255 | 3.0682 | 41.3814 | 3.0660 |
| 400 | 43.0425 | 3.0002 | 43.1077 | 2.9977 |
| 500 | 45.6696 | 2.8976 | 45.7502 | 2.8945 |
| 600 | 49.7969 | 2.7419 | 49.9057 | 2.7379 |
| 700 | 56.8437 | 2.4993 | 57.0131 | 2.4939 |
| 800 | 71.1544 | 2.1145 | 71.4929 | 2.1071 |

***Table 3:*** The comparison of the results by two approaches.
($B = 2*10^{-5}$, $b/H = 10^{-2}$, $Fr_F = 1.56*10^{-7}$).

This general form is because of neglecting the viscous and diffusional terms in the oscillating system (the absence of boundary effects). This conclusion is also in agreement with the asymptotic relation in thermo-vibrational problem in a cylindrical geometry [26].

It was emphasized in the previous section (Figs. (4) and (7)) that the critical Rayleigh numbers for sub-harmonic mode tend to asymptotic values. This fact can be explained by considering the stable points in Mathieu's stability diagram. For example, for the case of a sub-harmonic solution (with dimensionless frequency $\omega^*/2$) in which the layer is heated from below it is found that $A \to 0$ and $Q \to \pm 0.9$. For this case, the following asymptotic relationship is found (for $\frac{k_c^2}{k_c^2 + \pi^2} \to 1$):

$$Ra_{T_c} \approx 0.445 \frac{B}{\delta^* Fr_F} \cdot \text{ (Sub-harmonic, layer heated from below)} \tag{46}$$

For the other case corresponding to sub-harmonic modes for the layer heated from above a similar relation is found ($k_c^2/(k_c^2 + \pi^2) \to 1$):

$$Ra_{Tc} \approx -0.445 \frac{B}{\delta^* Fr_F} \cdot \text{ (Sub-harmonic, layer heated from above)} \tag{47}$$

Examining eqs. (46) and (47), we conclude that for the onset of convection i.e. gravitational effect has no effect.

For the layer heated from above, following the same reasoning we find:

$$Ra_{Tc} \approx -3.75 \frac{B}{\delta^* Fr_F} \tag{48}$$

Relation (48) is valid for super-harmonic case ( $A \to 0$, $Q \to -7.5$).

Some discussions at this stage is necessary, regarding the asymptotic formulas. Gresho and Sani [6] gave an empirical relation for sub-harmonic mode. However, the computations of Clever et.al [9] shows that this is not valid all the time. The results of Aniss et al. [25] shows the existence of asymptotic behavior, however they did not provide any relation. So, not only numerically we showed the existence of this behavior, we provided a mathematical basis for it, too. In addition, we proposed other asymptotic relation for the super-harmonic mode.

### *Weakly nonlinear stability analysis*

In the previous section, we studied the linear stability analysis of the conductive solution. The aim of this section is to obtain the amplitude equation, which is left undetermined by linear analysis. This analysis allows us to determine the characteristics of solutions near the bifurcation point. Our method is based on multi-scale approach. The nonlinear stability problem of time-averaged formulation is expressed in terms of ($\psi$, $\theta$, $F$) as follow:

$$\frac{\partial}{\partial t}\begin{bmatrix} B\nabla^2\psi \\ \theta \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -\nabla^2 & -Ra_T\dfrac{\partial}{\partial x^*} & -Ra_v\dfrac{\partial^2}{\partial x^{*2}} \\ -\dfrac{\partial}{\partial x^*} & \nabla^2 & 0 \\ 0 & \dfrac{\partial}{\partial x^*} & \nabla^2 \end{bmatrix}}_{L} \begin{bmatrix} \psi \\ \theta \\ F \end{bmatrix} + \begin{bmatrix} N_1 \\ N_2 \\ 0 \end{bmatrix} \tag{49}$$

in which **L** represents a linear operator where as $N_1$ and $N_2$ are nonlinear operators:

$$N_1 = -Ra_v\left[\frac{\partial^2\theta}{\partial x^{*2}}\frac{\partial F}{\partial z^*} + \frac{\partial\theta}{\partial x^*}\frac{\partial^2 F}{\partial x^*\partial z^*} - \frac{\partial^2 F}{\partial x^{*2}}\frac{\partial\theta}{\partial z^*} - \frac{\partial F}{\partial x^*}\frac{\partial^2\theta}{\partial x^*\partial z^*}\right],$$
$$N_2 = \frac{\partial\psi}{\partial x^*}\frac{\partial\theta}{\partial z^*} - \frac{\partial\psi}{\partial z^*}\frac{\partial\theta}{\partial x^*}.$$

In order to study the onset of thermo-vibrational convection near the critical thermal Rayleigh number, the linear operator and the solution are expanded into power series of the positive small parameter $\eta$, defined by:

$$Ra_T = Ra_{Tc} + \eta\, Ra_{T1} + \eta^2\, Ra_{T2} + \cdots \tag{50}$$

Thus:

$$\begin{aligned} [\psi,\theta,F] &= \eta_2[\psi_1,\theta_1,F_1] + \eta^2[\psi_2,\theta_2,F_2] + \cdots \\ \mathbf{L} &= \mathbf{L}_0 + \eta\mathbf{L}_1 + \eta^2\mathbf{L}_2 + \cdots \end{aligned} \tag{51}$$

where:

$$\mathbf{L}_0 = \begin{bmatrix} -\nabla^2 & -Ra_{Tc}\dfrac{\partial}{\partial x^*} & -\dfrac{(\delta^* Fr_F\omega^*)^2}{2B}Ra_{Tc}^2\dfrac{\partial^2}{\partial x^{*2}} \\ -\dfrac{\partial}{\partial x^*} & \nabla^2 & 0 \\ 0 & \dfrac{\partial}{\partial x^*} & \nabla^2 \end{bmatrix},$$

$$\mathbf{L}_1 = \begin{bmatrix} 0 & -Ra_{T1}\frac{\partial}{\partial x^*} & -\frac{(\delta^* Fr_F \omega^*)^2}{B} Ra_{T1} Ra_{Tc} \frac{\partial^2}{\partial x^{*2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

($L_0$ is the operator which governs the linear stability). In above operators $Ra_v$ is also expanded:

$$Ra_v = \frac{(\delta^* Fr_F \omega^*)^2}{2B}\left[Ra_{Tc}^2 + 2\eta Ra_{T1} Ra_{Tc} + \eta^2 (2Ra_{Tc} Ra_{T2} + Ra_{T1}^2) + \cdots\right] \quad (52)$$

By replacing (50-52) in (49), and after introducing the classical time transformation as:

$$\frac{\partial}{\partial t^*} = \eta \frac{\partial}{\partial t_1^*} + \eta^2 \frac{\partial}{\partial t_2^*} + \cdots$$

which on equating the same power of $\eta$ leads us to a sequential system of equations.

At each order of $\eta$, a linear eigenvalue problem is recovered. At the first order ($\eta$) the perturbation is written in the following form:

$$\begin{bmatrix} \psi_1 \\ \theta_1 \\ F_1 \end{bmatrix} = S\,(t_1^*, t_2^*, \ldots) \begin{bmatrix} (\pi^2 + k^2)/k^2 \sin \pi z^* \sin kx^* \\ -(\pi^2 + k^2)/k \sin \pi z^* \cos kx^* \\ \sin \pi z^* \sin kx^* \end{bmatrix}.$$

The amplitude $A$ depends on slow times ($t_1^*$, $t_2^*$,...).

At the second order $\eta^2$, the existence of convective solution requires that the solvability lemma to be satisfied, in other words there must be a non zero solution for the adjoint of $L_0$ which is defined by:

$$\mathbf{L}_0^* = \begin{bmatrix} -\nabla^2 & \frac{\partial}{\partial x^*} & 0 \\ Ra_{Tc}\frac{\partial}{\partial x^*} & \nabla^2 & -\frac{\partial}{\partial x^*} \\ -\frac{(\delta^* Fr_F \omega^*)^2}{2B} Ra_{Tc}^2 & 0 & \nabla^2 \end{bmatrix}.$$

associated with identical boundary conditions. From adjoint operator, we obtain:

$$Ra_{Tc}^* = Ra_{Tc}.$$

Also, we find $Ra_{T1}$=0 and amplitude $A$ does not depend on time scale $t_1^*$.

At the third order $\eta^3$ by invoking solvability condition and Fredholm alternative we obtain the amplitude equation:

$$\frac{dS}{dt_2^*} = \alpha (S - \beta S^3). \quad (53)$$

in which α and β are defined as:

$$\alpha = \frac{k^2}{(k^2+\pi^2)^2}\left[(k^2+\pi^2) - \frac{(\delta^* Fr_F \omega^*)^2}{B} k^2 Ra_{Tc}\right] Ra_{T2},$$

$$\beta = \frac{(\pi^2+k^2)^2\left[1 - \dfrac{k^4(\delta^* Fr_F \omega^*)^2}{B(\pi^2+k^2)^3} Ra_{Tc}^2\right]}{8Ra_{T2}\left[(\pi^2+k^2) - \dfrac{(\delta^* Fr_F \omega^*)^2}{B} k^2 Ra_{Tc}\right]}.$$

in $\alpha$, $\beta$ $Ra_{T2}$ is defined as $Ra_{T2} = (Ra_T - Ra_{Tc})/\eta^2$ which is the control parameter . When there is no vibrational effect, the amplitude of thermo-convective flow near the bifurcation point is proportional to:

$$S \approx \sqrt{Ra_T - Ra_{Tc}}.$$

this result is in agreement with [16].

Under the effect of vibration $\alpha$, $\beta$ are both positive which results in a supercritical pitchfork bifurcation.

The averaged Nusselt number can be expressed as:

$$\overline{Nu} = 1 + \frac{\Lambda}{4\pi^2}(Ra_T - Ra_{Tc}) \tag{54}$$

Fig. 9, illustrates the value of the coefficient Λ in terms of $k_c$. It is evident that vibration in the limiting case of high-frequency and small amplitude, due to the reduction of the wave number, decreases the heat-transfer rate.

## Conclusions

The main conclusions of this paper regarding the onset and finite amplitude convection under the influence of mechanical vibration are as follows:

i) As the nature of the thermo-vibrational problem is quite different from that of classical gravity induced natural convection due to the existence of different time scales and amplitude ratios, a detailed scale analysis method is performed. This order of magnitude analysis helped the finding of the relations between relevant time scales and amplitude ratios under high- frequency and small-amplitude vibration. For the first time, the terms “high” and “small” are fully explained in the context of high-frequency and small amplitude vibrations in porous media.

ii) It is shown that, out of numerous possible combinations between frequency and amplitude of vibration, high frequency and small amplitude constitutes an option to control the onset of convection. Interestingly, this limit is compatible to the level of residual acceleration on the Space Station [28]. Analytical relations were obtained from which one may easily calculate the critical Rayleigh and wave numbers for this situation. In addition these relations were generalized (Hele-Shaw configuration). It is shown that the response of the system in this case is always synchronous (the same frequency as of external vibration). Limiting value of frequency was predicted beyond which the time-averaged method is not valid.

iii) In the context of the so-called direct method, it was shown that the stability analysis leads to the study of the second order differential equation with periodic coefficients. It was shown under which conditions harmonic and sub-harmonic responses may be observed. The existence of super-harmonic response for the layer heated from above has been discussed. In a novel approach, we established a systematic procedure through a set of criteria to compare the stability results obtained by time-averaged and direct methods. This procedure is general and can be applied to any geometry.

iv) Our study bridges the existing gap between two schools of thoughts on thermal stability analysis of thermo-vibrational problem. It fully shows the shortcomings and advantages of each method.

v) The thermal Rayleigh numbers tend to an asymptotic value for the onset of convection at higher frequencies for the sub-harmonic and super-harmonic solution. Analytical expressions for these modes have been presented. We compared qualitatively our results with what have been published in the literature concerning fluid media. Some anomalies in the literature were discussed.

The unexpected conclusion is that the onset of convection under micro-gravity conditions in the range of high frequency has a sub-harmonic response.

vi) The weakly non-linear analysis of time-averaged equations reveals that the bifurcation is of super-critical pitchfork type. An expression for Nusselt number for the thermovibrational problem in porous media has been found, it is illustrated that vibration decreases heat transfer rate.

## Acknowledgements

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## References

1. DA Nield and A Bejan. "Convection in porous media". 2nd edition. Springer, Berlin (1999).
2. DB Ingham and I Pop. "Transport phenomena in porous media". Elsevier science Ltd (1998).
3. K Vafai. "Handbook of porous media". Marcel and Decker (2000).
4. JID Alexander. "Residual gravity jitter effects on fluid processes". Microgravity sci. technol 7 (1994): 131-136.
5. B Feuerbacher, H Hamacher and R Jilg. "Compability of microgravity experiments with spacecraft disturbances". Z. Fluwiss. Weltraumforsch 12 (1988): 145.
6. PM Gresho and RL Sani. "The effects of gravity modulation on the stability of a heated fluid layer". J. Fluid Mech 40 (1970): 783.
7. GZ Gershuni, EM Zhukovitskii and IuS Iurkov. "On convective stability in the presence of periodically varying parameter". J. Applied Math. And Mech 34 (1970): 442.
8. GZ Gershuni and EM Zhukovitskii. "Convective stability of incompressible fluids". Keter Publications (1976).
9. R Clever, G Schubert and FH Busse. "Two-dimensional oscillatory convection in a gravitationally modulated fluid layer". J. Fluid Mech 253 (1993): 663.
10. GZ Gershuni and DU Lyubimov. "Thermal vibrational convection". Wiley (1998).
11. SM Zenkovskaya. "Action of high-frequency vibration on filtration convection". J. Applied Mech. and Tech. Phys 32 (1992): 83.
12. SM Zenkovskaya and TN Rogovenko. "Filtration convection in a high-frequency vibration field". J. Applied Mech. and Tech. Phys 40 (1999): 379.
13. G Bardan and A Mojtabi. "On the Horton-Rogers-Lapwood convective instability with vertical vibration". Phy.Fluids 12 (2000): 2723.
14. H Khallouf, GZ Gershuni and A Mojtabi. "Some properties of convective oscillations in porous medium". Numer. Heat Transfer, Part A 30 (1996): 605.
15. IB Simonenko and SM Zenkovskaya. "On the effect of high frequency vibration on the origin of convection". Izv.Akad.Nauk SSSR, Mekh.Zhidk.Gaza 5 (1966): 51.
16. IB Simonenko. "A justification of averaging method for a problem of convection in a field rapidly oscillating forces and other parabolic equations". Math.USSR Sbornik 16 (1972): 245.
17. A Bejan. "Convection heat transfer". 2nd edition. Wiley, New York (1994).
18. A Bejan. "Simple methods for convection in porous media: scale analysis and the intersection of asymptotes". Int. J. Energy Res 27 (2003): 859.

19. SH Davis. “The stability of time-periodic flows”. Ann. Rev. fluid mech 8 (1976): 57.
20. NW Mc Lachlan. “Theory and application of Mathieu Functions”. Dover, New York (1964).
21. WJ Cunningham. “Introduction to nonlinear analysis”. McGraw-Hill, New York (1958).
22. DW Jordan and P Smith. “Nonlinear ordinary differential equations”. Oxford, (1987).
23. a. L Ruby. “Applications of the Mathieu equation”. Am. J. Phy 24 (1996): 39.
    b. KH Rosen. “Discrete Mathematics and Its Applications”. McGraw-Hill Higher Education, Eighth Edition (2018).
24. FM Phelps and JH Hunter. “An analytical solution of the inverted pendulum”. Am. J. Phy 33 (1965): 285.
25. a. S Aniss, M Souhar and M Belhaq. “Asymptotic study of the convective parametric instability in Hele-Shaw cell”. Phys. Fluids 12 (2000): 262.
    b. S Saravanan and A Arunkumar. “Convective instability in a gravity modulated anisotropic thermally stable porous medium”. Int. Journal of Eng. Science 48 (2010): 742.
26. M Wadih and B Roux. “Natural convection in a long vertical cylinder under gravity modulation”. J. Fluid Mech 193 (1988): 391.
27. E Palm, JE Weber and O Kvernold. “On steady convection in a porous medium”. J. Fluid Mech 54 (1972): 153.
28. ES Nelson. “An examination of anticipated g-jitter on space station and its effects on materials processes”. NASA TM 103775 (1994).

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